Representations of the symmetric group and its Hecke algebra

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Abstract

This paper is a survey on the representation theory of the symmetric group and its Hecke algebra in arbitrary characteristic.

1 Introduction

Let $n \in \mathbb{N}$ and let $\mathfrak{S}_n$ be the symmetric group acting on the left on $\{1, 2, \ldots, n\}$. Let k be a field and consider the group algebra $k\mathfrak{S}_n$. This is the k-algebra with

- k-basis $\{t_\sigma \mid \sigma \in \mathfrak{S}_n\}$,
- multiplication: for all $(\sigma, \sigma') \in \mathfrak{S}_n^2$ we have $t_\sigma . t_{\sigma'} = t_{\sigma\sigma'}$.

The aim of this course is to study the *Representation Theory* of $\mathfrak{S}_n$ over k. A representation of $\mathfrak{S}_n$ over k is a finite dimensional k-vector space V together with a morphism:

$$\rho : \mathfrak{S}_n \to \mathrm{End}_k(V).$$

The datum of such a representation is equivalent to the datum of the (left) $k\mathfrak{S}_n$-module where the action of $x \in k\mathfrak{S}_n$ on all $v \in V$ is given by $x.v := \rho(x)(v) \in V$ (so we will freely pass from one term to another).

We will focus in particular on the *irreducible representations* or equivalently on the *simple $k\mathfrak{S}_n$-modules*. They are the $k\mathfrak{S}_n$-modules which do not admit any non trivial submodule (that is different from $\{0\}$ and themselves). The *dimension* of the associated representation is then by definition the dimension of V as a k-vector space.

The main questions we will address are quite natural:

- We want to find all the simple $k\mathfrak{S}_n$-modules for all fields k (that is find the number of them, a natural labeling for them, a way to construct them explicitly etc.)
- We want to compute the dimensions of these simple $k\mathfrak{S}_n$-modules.

In characteristic 0, these questions have been solved by Frobenius in the beginning of the twentieth century. However, when k is a field of characteristic $p > 0$, it is a remarkable fact that these problems are still open in most cases !

Actually, instead of strictly considering this problem, we will slightly generalize it by addressing the same question for a generalization of the group algebra of the symmetric group: its Iwahori-Hecke algebra. In fact, we will see that the representations of the group algebra and the representations of its Hecke algebra admit close relations which will help to provide (partial and sometimes conjectural) answers to the above problems.

The survey will be organized as follows. The aim of the first part is to introduce the Iwahori-Hecke algebra of the symmetric group. Several properties on the symmetric group will be needed here. We then begin the study of the representation theory of this algebra by giving an overview of Kazhdan-Lusztig Theory. In the fourth part, using this theory, we study the simple modules for Hecke algebras in the so called semisimple case which is the easier case to consider. We then begin the study of the most difficult case: the non semisimple case. We introduce a very important object in this settings: the decomposition matrix, and give a conjectural connection between the representation theory of the symmetric group and the one of its Iwahori-Hecke algebra using this object. Then, we give an introduction to Ariki-Lascoux-Lecrec-Thibon's Theory. The fifth part ends with Ariki's Theorem which gives a way to compute these decomposition matrices and the dimensions of the simple modules for Hecke algebras in characteristic 0. Finally we give an introduction to some recent results on the graded representation theory of these algebras.

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2 The symmetric group and its Hecke algebra

In this section, we give several combinatorial properties for the symmetric group which will help us in the following section. The section ends with our "generalization" of our main object which is the Hecke algebra of $\mathfrak{S}_n$. Part of this section is inspired by the book of Mathas [18, Ch. 1] which is a very nice (and far more complete) reference on this subject.

For $i = 1, \dots, n-1$, we set s_i to be the transposition $(i, i+1)$. Then, it is known that $\mathfrak{S}_n$ admits a presentation by

- generators: $s_1, s_2, \dots, s_{n-1}$
- relations:

$$\begin{array}{ccccc} s_i^2 & = & 1 & \text{for } i = 1, 2, \dots, n-1 \\ s_i s_j & = & s_j s_i & \text{for } 1 \leq i < j-1 \leq n-2 \\ s_i s_{i+1} s_i & = & s_{i+1} s_i s_{i+1} & \text{for } i = 1, 2, \dots, n-2 \end{array}$$

The two last relations are known as braid relations.

Remark 2.1. This presentation makes $\mathfrak{S}_n$ into a *finite Coxeter group*. Such a group admits a presentation by

- generators: $r_1, r_2, \dots, r_{n-1}$
- relations $(r_i r_j)^{m_{ij}} = 1$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for $i \neq j$ (and $m_{ij} = \infty$ means no relation is imposed). There is a classification of the (finite) Coxeter groups in terms of Dynkin diagrams. They contain the set of Weyl groups as a special case.

Remark 2.2. The finite Coxeter groups (and thus the symmetric group) have a concrete realization in terms of real reflection groups. Such groups are generated by a set of reflections in a finite dimensional Euclidean space. There is also a generalization of these groups: the complex reflection groups which are the finite groups generated by pseudo-reflection (a pseudo reflection being a non trivial invertible element which acts trivially on a hyperplane of a finite dimensional complex vector space). These groups have also been classified by Shephard and Todd, see [2] for a study of these groups.

As already noted in the introduction, our main problem will be the following one

Main Problem: Determine all simple $k\mathfrak{S}_n$ modules for all fields k.

Before doing representation theory, we need to study the form of the elements of the symmetric group in terms of the above presentation. So let $w \in \mathfrak{S}_n$ and write $w = s_{i_1} s_{i_2} \dots s_{i_r}$ for $(i_1, \dots, i_r) \in \{1, \dots, n-1\}^r$.

If r is minimal then we say that this expression for w is *reduced* and that w is of length r. We denote $l(w) = r$ and, by this, we have defined a *length function*:

$$l : \mathfrak{S}_n \to \mathbb{N}$$

One can see that for all $i \in \{1, \dots, n-1\}$, we have $l(s_i w) = l(w) \pm 1$.

Of course, there are several possible reduced expressions for the same element. For example the element $(1,2)(1,3)$ in $\mathfrak{S}_3$ can be written $s_1 s_2 s_1$ or $s_2 s_1 s_2$. The following theorem which can be proved combinatorially shows what is the connection between all these reduced expressions (see [18, Theorem 1.8]).

Theorem 2.3 (Matsumoto)**.** *Let $w \in \mathfrak{S}_n$ and let $w = s_{i_1} s_{i_2} \dots s_{i_r}$ be a reduced expression for w. Then $s_{j_1} s_{j_2} \dots s_{j_r}$ is a reduced expression for w if and only if one can transform this expression to $s_{i_1} s_{i_2} \dots s_{i_r}$ using only the braid relations.*

Given elements $y \in W$ and $w \in W$, we write $y \leq w$ if y can be obtained by omitting some terms in the reduced expressions for W. The resulting partial order on W is called the *Bruhat-Chevalley order*. For example we have in $\mathfrak{S}_3$:

$$1 \leq s_1 \leq s_1 s_2 \leq s_1 s_2 s_1, \text{ and } 1 \leq s_2 \leq s_1 s_2 \leq s_1 s_2 s_1$$

Using this Theorem, we will be ready to define the Iwahori-Hecke algebra (or simply the Hecke algebra) of the symmetric group. Let A be an integral domain with 1 and let v be an invertible element in A and set $u = v^2$. The definition is based on the presentation of the symmetric group.

Definition 2.4. The *Hecke algebra $\mathcal{H}_A(v)$* of $\mathfrak{S}_n$ is the associative A-algebra with unit defined by

- generators: $T_{s_1}, T_{s_2}, \dots, T_{s_{n-1}}$
- relations:

$$\begin{array}{rcll} (T_{s_i} - u)(T_{s_i} + 1) & = & 0 & \text{for } i = 1, 2, \dots, n-1 \\ T_{s_i} T_{s_j} & = & T_{s_j} T_{s_i} & \text{for } 1 \leq i < j - 1 \leq n-2 \\ T_{s_i} T_{s_{i+1}} T_{s_i} & = & T_{s_{i+1}} T_{s_i} T_{s_{i+1}} & \text{for } i = 1, 2, \dots, n-2 \end{array}$$

Note that if we set $u = 1$, the relations are exactly the one given in the presentation of the symmetric group. Thus the Hecke algebra is nothing but the group algebra of $A\mathfrak{S}_n$ in this special case. This is the reason why this algebra can be seen as a deformation (or as a generalization) of the symmetric group. So instead of finding an answer to our problem, we have generalized it:

New main Problem Determine all simple $\mathcal{H}_k(q)$ modules for all field k and parameter $q \in k^\times$.

Remark 2.5. By the works of Iwahori, this algebra also naturally appears in another situation. Assume that q is a prime power and consider the general linear group $G := \mathrm{GL}_n(\mathbb{F}_q)$ over the field $\mathbb{F}_q$. Let B be a the subgroup of upper triangular matrix in G. Let $\mathrm{Ind}_B^G(1)$ be the induced representation of the trivial one then the algebra $\mathrm{End}_{kG}(\mathrm{Ind}_B^G(1))$ is isomorphic to $\mathcal{H}_k(q)$. In particular the representation theory of the Hecke algebra is closely connected with the representation theory of G. We refer to [11] for a study of these connections (see also [12, §8.4].)

We now want to produce a basis for our algebra. This basis should be a natural generalization of the natural basis for our group algebra. For $w \in \mathfrak{S}_n$, we consider a reduced expression :

$$w = s_{i_1} s_{i_2} \dots s_{i_r}$$

We then denote

$$T_w := T_{s_{i_1}} T_{s_{i_2}} \dots T_{s_{i_r}}.$$

A priori, we need to show that this is well defined and does not depend on the choice of a reduced expression for w. But this fact just follows from Theorem 2.3 ! With this notation, the identity element of the algebra is just T_w with w the identity element. We denote it by 1. The next step consists in showing that these element form a basis of our algebra. To do this, we first find a multiplication formula between two such elements. This is given as follows. For $i \in \{1, \dots, n-1\}$ and $w \in \mathfrak{S}_n$, we have:

$$T_{s_i} T_w = \begin{cases} T_{s_i w} & \text{if } l(s_i w) > l(w) \\ u T_{s_i w} + (u-1) T_w & \text{if } l(s_i w) < l(w) \end{cases}$$

Indeed, the first case just follows from the definition of reduced expression. In the second case, as we have $l(s_i w) = l(w) - 1$, one can see that w has a reduced expression beginning by s_i. Indeed, we have a reduced expression of $s_i w$ of the form $s_{j_1} \dots s_{j_s}$ and thus the expression $s_i s_{j_1} \dots s_{j_s}$ is a reduced expression for w (because $s_i^2 = 1$). The result follows then by an easy computation. One can also note that these elements are all invertible. This comes from the fact that each T_{s_i} is invertible with inverse $u^{-1} T_{s_i} - 1 + u^{-1}$.

Theorem 2.6. *$\mathcal{H}_A(v)$ is free as an A-module with an A-basis given by the set*

$$\{T_w \mid w \in \mathfrak{S}_n\}$$

In particular, the dimension of $\mathcal{H}_A(v)$ is $n!$.

Proof. We give the strategy of the proof. First, the above multiplication formula already shows that our set is a generating set for the A-module $\mathcal{H}_A(v)$. So the hard part is to show that this is A-free. This is done as follows. Let E be the free A-module with a basis indexed by the elements of $\mathfrak{S}_n$:

$$\{e_w \mid w \in \mathfrak{S}_n\}$$

From this, we construct an algebra which enjoys similar relations as in the Hecke algebra. This is the subalgebra denoted by $\mathcal{A}$ of $\mathrm{End}_A(E)$ generated by operators θ_i with $i \in \{1, \dots, n-1\}$ which are defined as follows:

$$\theta_i(e_w) = \begin{cases} e_{s_i w} & \text{if } l(s_i w) > l(w) \\ u e_{s_i w} + (u-1) e_w & \text{if } l(s_i w) > l(w) \end{cases}$$

The aim of the rest of the proof is to show that there is a surjective morphism

$$\Psi : \mathcal{H}_A(u) \to \mathcal{A}$$

sending each T_{s_i} to θ_i. This is sufficient to prove our Theorem. Indeed, assume that we have elements aw in A such that

$$\sum_{w \in \mathfrak{S}_n} a_w T_w = 0$$

Applying Ψ and evaluating at e_1 leads to

$$\sum_{w \in \mathfrak{S}_n} a_w e_w = 0$$

and this show that Ψ is an isomorphism and that our set is A-free. Now to prove the existence of Ψ, all we have to do is to use the presentation of the Hecke algebra by showing that the θ_i's satisfy the relations of the Hecke algebra. This is done by simple (but quite long) computations. □

Remark 2.7. All the above results admit generalizations to the case of Coxeter groups. However, the situation is more complicated in the case of complex reflection groups .

Remark 2.8. Using the presentation of our algebra, one can already construct the "linear" representations of $\mathcal{H}_A(v)$, that is the representations of dimension 1. Such representations are of the form

$$\rho : \mathcal{H}_A(v) \to \mathrm{Mat}_{1,1}(A) = A$$

such that $(\rho(T_{s_i}) - u)(\rho(T_{s_i}) + 1) = 0$ for all $i = 1, \ldots, n-1$. Thus we have $\rho(T_{s_i}) = u = v^2$ or $\rho(T_{s_i}) = -1$. Thus, for all $w \in \mathfrak{S}_n$, we obtain:

$$\rho(T_w) = u^{l(w)} \qquad \text{or} \qquad \rho(T_w) = (-1)^{l(w)}$$

These representations may be seen as analogues of the trivial and the sign representations for the symmetric group. We will come back later to this point.

3 Kazhdan-Lusztig Theory

The Kazhdan-Lusztig Theory is a powerful and deep theory which has been described for the first time in [7], again in the wider context of finite Coxeter group. This results imply in particular the existence of a new basis, the Kazhdan-Lusztig basis, which will be particularly well adapted to the study of the representation theory of the Hecke algebra.

Before giving the definition of this basis, let us first construct a small variation of our "standard basis". In all this section, we assume that v is an indeterminate and that $A := \mathbb{Z}[v, v^{-1}]$. For all $w \in \mathfrak{S}_n$, we set $\widetilde{T}_w := v^{-l(w)} T_w$. Then, the set

$$\{\widetilde{T}_w \mid w \in \mathfrak{S}_n\}$$

is a basis of $\mathcal{H}_A(v)$ (with $\widetilde{T}_1 = T_1 = 1$).

It is an easy exercise to show the following proposition:

Proposition 3.1 (Kazhdan-Lusztig)**.** *For $a = \sum_{i \in \mathbb{Z}} a_i v^i \in A$ with $a_i \in \mathbb{Z}$ for all $i \in \mathbb{Z}$ (and all but a finite number are non zero), we set $\overline{a} := \sum_{i \in \mathbb{Z}} a_i v^{-i} \in A$. Then the map:*

$$\begin{array}{cccc} - : & \mathcal{H}_A(v) & \to & \mathcal{H}_A(v) \\ & h = \sum_{w \in W} a_w \widetilde{T}_w & \mapsto & \overline{h} := \sum_{w \in W} \overline{a_w} \widetilde{T}_{w^{-1}}^{-1} \end{array}$$

is a ring automorphism.

The definition of the Kazhdan-Lusztig basis is then given by the following theorem which is stated without proof.

Theorem 3.2 (Kazhdan-Lusztig)**.** *For all $w \in \mathfrak{S}_n$ there exists a unique element $C_w \in \mathcal{H}_A(v)$ such that*

$$\overline{C_w} = C_w \text{ and } C_w = \widetilde{T}_w + \sum_{x \in W} a_x \widetilde{T}_x$$

where $a_x \in v^{-1}\mathbb{Z}[v^{-1}]$ if $x \neq w$ and $a_x = 0$ unless $x \leq w$. The set

$$\{C_w \mid w \in \mathfrak{S}_n\}$$

is an A-basis of $\mathcal{H}_A(v)$ called the Kazhdan-Lusztig basis (with $C_1 = T_1 = 1$).

We refer to [7] for a proof of this theorem which can be done using only elementary methods by induction. There is a recursive way to compute this basis as well as formulae for multiplication between this basis and the "standard" one.

Example 3.3. Assume that $n = 3$. Then we have

$$\mathfrak{S}_3 = \{1, s_1, s_2s_1, s_2, s_1s_2, s_1s_2s_1\}$$

One can check that the Kazhdan-Lustig basis is here given by:

- $C_1 = T_1$,

- $C_{s_2} = \widetilde{T}_{s_2} + v^{-1}T_1$, $C_{s_1} = \widetilde{T}_{s_1} + v^{-1}T_1$
- $C_{s_2s_1} = \widetilde{T}_{s_2s_1} + v^{-1}\widetilde{T}_{s_1} + v^{-1}\widetilde{T}_{s_2} + v^{-2}T_1$, $C_{s_1s_2} = \widetilde{T}_{s_1s_2} + v^{-1}\widetilde{T}_{s_2} + v^{-1}\widetilde{T}_{s_1} + v^{-2}T_1$
- $C_{s_1s_2s_1} = \widetilde{T}_{s_1s_2s_1} + v^{-2}\widetilde{T}_{s_2} + v^{-2}\widetilde{T}_{s_1} + v^{-1}\widetilde{T}_{s_1s_2} + v^{-1}\widetilde{T}_{s_2s_1} + v^{-3}T_1$

The definition of this basis allows the definition of Kazhdan-Lusztig cells which gives a way to decompose the Weyl group into smaller pieces. We will see that these pieces admit some important properties.

Definition 3.4. Let y and w in $\mathfrak{S}_n$. Then we write $y \leq_{L,s_i} w$ if C_y appears in the expansion of $C_{s_i}C_w$ as a linear combination of the Kazhdan-Lusztig basis with a non zero coefficient. We write $y \leq_L w$ if and only if there exists a sequence of elements in $\mathfrak{S}_n$ such that

$$y = y_0 \leq_{L,s_{i_0}} y_1 \leq_{L,s_{i_1}} y_2 \leq_{L,s_{i_0}} \ldots \leq_{L,s_{i_k}} y_{k+1} = w$$

We denote $y \simeq_L w$ if and only if $w \leq_L y$ and $y \leq_L w$. The relation $\simeq_L$ is an equivalence relation and the equivalence classes are called the Kazhdan-Lusztig cells. We have a partition:

$$\mathfrak{S}_n = \bigsqcup_{\Gamma \text{ KL cells}} \Gamma$$

Example 3.5. For all $w \in \mathfrak{S}_n$, we see that we have $w \leq_L 1$.

Example 3.6. Assume that $n = 3$. The group is given by:

$$\mathfrak{S}_3 = \{1, s_1, s_2, s_2s_1, s_1s_2, s_1s_2s_1\}$$

The left cells are given by:

$$\{1\}, \{s_1, s_2s_1\}, \{s_2, s_1s_2\}, \{s_1s_2s_1\}$$

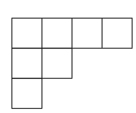
Is there an easy combinatorial way to find these cells ? the answer is yes: this is the Robinson-Schensted correspondence which is given as follows (We refer to [9] for a more detailed review of this correspondence). This correspondence involves a number of combinatorial objects which naturally appear in the representation theory of the symmetric group.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a *partition* of rank n, that is, a sequence of positive integers of total sum n such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. The *Young diagram* of λ is the set

$$[\lambda] = \{(i, j) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0} \mid 1 \leq j \leq \lambda_i\}.$$

The elements of λ are the nodes of λ. The Young diagram is usually represented as an array of boxes as in the following example:

Example 3.7. The Young diagram of the partition (4.2.1) of 7 is



A *standard Young tableau* of form λ is the Young diagram $[\lambda]$ where each node is filled with one integer in $\{1, \ldots, n\}$. In addition:

- each integer in $\{1, \ldots, n\}$ appear exactly once in the Young tableau.
- The entries increase from left to right and top to bottom.

Example 3.8. The standard Young tableaux of form (3.1) are:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}$$

The following is a standard Young tableau of form (4.2.2.1)

$$\begin{array}{|c|c|c|c|}\hline 1&3&4&8\\\hline 2&6\\\cline{1-2} 5&9\\\cline{1-2} \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}$$

Now we associate to each element $w \in \mathfrak{S}_n$ a pair of standard Young tableaux $(P(w), Q(w))$ with the same form $\lambda = (\lambda_1, \dots, \lambda_r)$. The procedure is done recursively as follows.

At the beginning, $P(w)$ and $Q(w)$ are empty Young tableaux. Then for all $i = 1, \dots, n$, we recursively add to $P(w)$ the integer $w(i)$ in order to have a standard tableau in the following way. If all the integers in the first row of $P(w)$ are less than $w(i)$, we add $w(i)$ at the right in the first row. Otherwise, we replace the smallest number j greater than $w(i)$ by $w(i)$ and we add j in the second row using the same procedure. At the end, we obtain a standard tableau with form λ.

In addition, we keep the record of the process by constructing a tableau $Q(w)$ of form λ. To do this, at each step, we add a box filled by i to the standard tableau at the place where a new box is created in $P(w)$. We obtain a standard tableau $Q(w)$ which gives the order of appearances of the nodes.

Example 3.9. Let $w = s_1 s_3 s_2 s_1 s_5$ in $\mathfrak{S}_6$. We construct the pair $(P(w), Q(w))$. We have $w(1) = 4$ so the first step gives:

$$\begin{array}{|c|}\hline 4\\\hline\end{array}\,,\; \begin{array}{|c|}\hline 1\\\hline\end{array}$$

Then we have $w(2) = 2$ and we obtain

$$\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\,,\; \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$$

we have $w(3) = 1$,

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}\,,\; \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

finally we have $w(4) = 3$, $w(5) = 6$ and $w(6) = 5$ which leads to the following pair:

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}$$

It is possible to show that we obtain a bijection

$$\begin{array}{ccc} \mathfrak{S}_n & \to & \bigcup_{\lambda \in \Pi_n} T_\lambda \times T_\lambda \\ w & \mapsto & (P(w), Q(w)) \end{array}$$

where T_λ denotes the set of all standard tableaux with form λ and Π_n the set of partitions of rank n.

Example 3.10. Here is the Robinson Schensted correspondence for $\mathfrak{S}_3$:

$w \in \mathfrak{S}_3$	$P(w)$	$Q(w)$
1	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline\end{array}$
s_1	$\begin{array}{ll}1&3\\2&\end{array}$	$\begin{array}{ll}1&3\\2&\end{array}$
s_2	$\begin{array}{ll}1&2\\3&\end{array}$	$\begin{array}{ll}1&2\\3&\end{array}$
s_1s_2	$\begin{array}{ll}1&3\\2&\end{array}$	$\begin{array}{ll}1&2\\3&\end{array}$
s_2s_1	$\begin{array}{ll}1&3\\3&\end{array}$	$\begin{array}{ll}1&3\\2&\end{array}$
$s_1s_2s_1$	$\begin{array}{l}1\\2\\3\end{array}$	$\begin{array}{l}1\\2\\3\end{array}$

We have the following combinatorial description of the cells which is given in the Kazhdan-Lusztig original article:

Proposition 3.11 (Kazhdan-Lusztig)**.** *Let w and y be two elements of the symmetric group. Then y and w lie in the same left cell if and only if $Q(w) = Q(y)$.*

We can check this fact in the examples above in $\mathfrak{S}_3$.

It turns out that one can construct some remarkable representations from these definitions of cells. To do this, let us fix Γ a left cell for $\mathfrak{S}_n$. We consider the ideal:

$$I_\Gamma^{\leq} = \langle C_w \mid w \in W,\ w \leq_L y \text{ for a } y \in \Gamma \rangle_A$$

and the ideal:

$$I_\Gamma^{<} = \langle C_w \mid w \in W,\ w \leq_L y \text{ for a } y \in \Gamma \text{ and } w \notin \Gamma \rangle_A$$

We then define the $\mathcal{H}_A(v)$-module $[\Gamma]_A := I_\Gamma^{\leq}/I_\Gamma^{<}$. By definition this is a free A-module with basis given by the classes of the Kazhdan-Lusztig basis indexed by the elements in the cell:

$$\{[C_w] \mid w \in \Gamma\}$$

The associated representation is called the left cell representation and one can show that two of them are isomorphic if and only if the form of the tableau in the Robinson Schensted correspondence is the same. So each isomorphism class of left cell representations is naturally labelled by a partition of rank n and reciprocally. We will denote it by ρ_λ. The associated cell module will be denoted by V^λ. Note that the dimension of this module as an A-module corresponds to the number of standard tableaux of form λ. We will actually see that in some cases, all the simple modules for the Hecke algebra are constructed in this way.

Example 3.12. Let us take $n = 3$. Assume that $w = 1$, then $\Gamma = \{w\}$ is a left cell of $\mathfrak{S}_3$ and the associated $\mathcal{H}_A(v)$-module has an A-basis $\{[C_1]\}$. We have $T_{s_1}.C_1 = v\widetilde{T}_{s_1}C_1 = v(C_{s_1} - v^{-1}T_1)C_1 = vC_{s_1} - C_1$. Thus, we obtain $T_{s_1}.[C_1] = -[C_1]$ and the same for T_{s_2}. The dimension of the module is 1 and it is the sign representation.

$$\begin{array}{rccc} \rho_{(3)}: & \mathcal{H}_A(v) & \to & \mathrm{Mat}_{1,1}(A) \\ & T_{s_1} & \mapsto & (-1) \\ & T_{s_2} & \mapsto & (-1) \end{array}$$

If we take $w = s_1s_2s_1$, we obtain the trivial representation which is given here by $T_{s_1}.[C_w] = v^2[C_w]$ and the same for T_{s_2}. Thus we have here two representations with dimension 1.

$$\begin{array}{rccc} \rho_{(1.1.1)}: & \mathcal{H}_A(v) & \to & \mathrm{Mat}_{1,1}(A) \\ & T_{s_1} & \mapsto & (v^2) \\ & T_{s_2} & \mapsto & (v^2) \end{array}$$

The cell module labelled by (2.1) has dimension 2. It is given by the from the left cell $\{s_1, s_1 s_2\}$ or the left cell $\{s_2, s_2 s_1\}$ (these two left cells give rise to equivalent representations). If we use the first cell, we have to compute $T_{s_1} C_{s_1}$ and $T_{s_1} C_{s_2 s_1}$. We obtain:

$$\begin{array}{rccc} \rho_{(2.1)}: & \mathcal{H}_A(v) & \to & \mathrm{Mat}_{2,2}(A) \\ & T_{s_1} & \mapsto & \begin{pmatrix} v^2 & 0 \\ v & -1 \end{pmatrix} \\ & T_{s_2} & \mapsto & \begin{pmatrix} -1 & v \\ 0 & v^2 \end{pmatrix} \end{array}$$

The above example can easily been generalized. The case $\lambda = (1^n)$ leads to the "trivial" representation and the case $\lambda = (n)$ to the "sign" representation.

4 Specialization and Semisimplicity

In the last section, we have only studied the case where $A := \mathbb{Z}[v, v^{-1}]$. In fact, this case is useful to find informations on the representation theory of the Hecke algebra over an arbitrary field. Let R be an arbitrary field and let $q \in R^\times$. As in the previous section assume that $A := \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate and set $u = v^2$. Then there exists a morphism:

$$\theta : A \to R$$

such that $\theta(v) = q$. Then we have:

$$\mathcal{H}_R(q) \simeq R \otimes_A \mathcal{H}_A(v)$$

We say that $\mathcal{H}_R(q)$ is obtained from $\mathcal{H}_A(v)$ by specialization. Thus, all Hecke algebras can be obtained from $\mathcal{H}_A(v)$ by this process of specialization. Note that the cell modules on the Hecke algebra $\mathcal{H}_A(v)$ give rise to well defined modules for the algebra $\mathcal{H}_R(q)$ using this process of specialization. Indeed, for all $\lambda \in \Pi_n$, we can define

$$V_R^\lambda := R \otimes_A V^\lambda.$$

They will also be called cell modules (associated to the specialization θ) from now on.

Assume now that k is a field and take $q \in k^\times$. Our aim is to find the set of simple $\mathcal{H}_k(q)$-modules $\mathrm{Irr}(\mathcal{H}_k(q))$. Let us first study a particular case: the case where the algebra $\mathcal{H}_k(q)$ is semisimple. This means that the algebra has a trivial Jacobson radical. In this case, any $\mathcal{H}_k(q)$-module can be written as a direct sum of simple $\mathcal{H}_k(q)$-modules. Kazhdan and Lusztig [7] (generalized by Graham and Lehrer using the concept of cellular algebras [14]) have proved:

Theorem 4.1 (Kazhdan-Lusztig)**.** *Assume that $\mathcal{H}_k(q)$ is a semisimple algebra. Then the cell modules V_k^λ form a complete set of non isomorphic simple $\mathcal{H}_k(q)$-modules. If $k = \mathbb{Q}(v)$ then the algebra $\mathcal{H}_k(v)$ is semisimple.*

Now it remains to know which cases are covered by the above Theorem. Let us first consider the case of the group algebra. Then by Maschke Theorem it is known that $k\mathfrak{S}_n$ is split semi-simple if the characteristic of k does not divide $n!$. Thus the above theorem implies that the cell modules give all the simple modules of the symmetric group in characteristic 0, for example

Corollary 4.2. *Consider the specialization $\theta : \mathbb{Z}[v, v^{-1}] \to \mathbb{Q}$ such that $\theta(v) = 1$ then*

$$\mathrm{Irr}(\mathbb{Q}\mathfrak{S}_n) = \{V_\mathbb{Q}^\lambda \mid \lambda \in \Pi_n\}.$$

Of course, there is an easy way to obtain these modules and to explicitly obtain the representations, see for example [10]. However, our method which involves the Hecke algebra, will also give us (at least conjectural) information in the case where the characteristic of k is positive.

Example 4.3. If we take Example 3.12, we see that, specializing v to 1 leads to the three well known non isomorphic irreducible representations of $\mathfrak{S}_3$.

First, note that an analogue of Maschke Theorem is more difficult to obtain for the Hecke algebra. Indeed, it can happen that this algebra is non semisimple even in characteristic 0 ! for example, in example 3.12, we see that the specialization $\theta : A \to \mathbb{Q}(\sqrt{-1})$ such that $\theta(v^2) = -1$ leads to two equivalent representations for the cell modules labelled by (3) and (1.1.1). One way to find a criterion of semisimplicity is to use the symmetric structure of the Hecke algebra. We briefly summarize this point of view (see [12, Ch. 7]):

- The Hecke algebra is equipped with a *symmetrizing trace*

$$\tau_v : \mathcal{H}_A(v) \to A$$

  This is an A-linear map satisfying $\tau(hh') = \tau(h'h)$ for all $(h, h') \in \mathcal{H}_A(q)$ and such that the associated bilinear form $(h, h') \mapsto \tau(hh')$ is non degenerate. In our case, the trace function is very simple. It is defined by its value on the standard basis of the Hecke algebra:

$$\tau_v(T_w) = \begin{cases} 1 & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases}$$

  In particular the group algebra of the symmetric group is symmetric. Actually, this is the case with any group algebra and the symmetrizing trace is a direct generalization of the one given above. The symmetric structure of an algebra is very useful and allows the generalization of many properties of the representation theory of groups to algebras (see [12, Ch. 7]).

- Consider the semisimple algebra $\mathcal{H}_{\mathbb{Q}(v)}(v)$. Then, extending the above trace leads to a symmetrizing trace function:

$$\tau_v : \mathcal{H}_{\mathbb{Q}(v)}(v) \to \mathbb{Q}(v).$$

  By the above theorem, we know all the simple modules for $\mathcal{H}_{\mathbb{Q}(v)}(v)$. To each representation $V^\lambda_{\mathbb{Q}(v)}$ of $\mathcal{H}_{\mathbb{Q}(v)}(v)$, one can associate its character which is the $\mathbb{Q}(v)$-linear map defined by $\chi^\lambda : \mathcal{H}_{\mathbb{Q}(v)}(v) \to \mathbb{Q}(v)$ such that $\chi^\lambda(h) = \text{trace}(\rho(h))$. This is a trace function and, as in the group case, one can show that these characters form a basis of the space of trace function in $\mathcal{H}_{\mathbb{Q}(v)}(v)$. As a consequence, there exist elements $d^\lambda(v)$ in $\mathbb{Q}(v)$ such that

$$\tau_v = \sum_{\lambda \in \Pi_n} d^\lambda(v)\chi^\lambda.$$

- It is a remarkable fact that the elements $d^\lambda(v)$ can be computed. In fact, one can show that they have the following form: $d^\lambda = 1/c^\lambda(v)$ where $c^\lambda(v) \in \mathbb{Z}[v, v^{-1}]$. These elements are called *Schur elements* and they play a powerful role in the representation theory of Hecke algebras. A nice criterion is then available to check the semisimplicty of the algebra (see [12, Thm. 7.4.7]. Assume that we have a specialization $\theta : A \to k$ such that $\theta(v) = q$. Then we have:

$$\mathcal{H}_k(q) \text{ is semisimple if and only if for all } \lambda \in \Pi_n \text{ we have } \theta(c^\lambda(v)) \neq 0.$$

- Using this one can show that in characteristic 0, $\mathcal{H}_k(q)$ is semisimple unless q is a root of unity. Indeed, these elements are the only root of the $d^\lambda(q)$'s.

Example 4.4. Take $n = 3$ then one can show that the elements $d^\lambda(v)$ are given as follows:

$$d^{(3)}(v) = (1/P).v^6,\ d^{(2.1)}(v) = (1/P).(v^4 + v^2),\ d^{(1.1.1)}(v) = 1/P$$

where $P = v^6 + 2v^4 + 2v^2 + 1$. One can check that

$$\tau_v = (1/P).(v^6\chi^{(3)} + (v^4 + v^2)\chi^{(2.1)} + \chi^{(1.1.1)}).$$

One can check this formula using the representations constructed in Example 3.12. Thus the Schur elements are:

$$c^{(3)}(v) = 1 + 2v^{-2} + 2v^{-4} + v^{-6},\ c^{(2.1)}(v) = v^2 + 1 + v^{-2},\ c^{(1.1.1)}(v) = v^6 + 2v^4 + 2v^2 + 1$$

Now we note that if $\theta : A \to k$ is a specialization such that $\theta(v) = q$ then

- If the characteristic of k is 0, we see that the specializations of the Schur elements are non zero if and only if $q^2 \neq -1$. Thus, $\mathcal{H}_k(q)$ is semisimple if and only if $\theta(v^2) \neq -1$.
- If the characteristic of k is p and if $q = 1$, we see that the specializations of the Schur elements are non zero if and only if $p \notin \{2,3\}$. Thus, $\mathcal{H}_k(q)$ is semisimple if and only if $p \notin \{2,3\}$ (and we recover Maschke Theorem).

Remark 4.5. Assume that $\mathcal{H}_k(q)$ and $\mathcal{H}_{k'}(q')$ are two semisimple Hecke algebras. Then the above theorem asserts that the cell modules give all the simple modules. Thus we have a natural bijection between the simple modules in both cases. This is the case for example when $k = \mathbb{Q}(v)$ and and $k' = \mathbb{Q}$ and $q' = 1$. In fact, the two algebras $\mathcal{H}_{\mathbb{Q}(v)}(v)$ and $\mathbb{Q}\mathfrak{S}_n$ are isomorphic. Lusztig has given an explicit morphism using the Kazhdan-Lusztig basis.

Example 4.6. Let us study the example of the group algebra. In this case, the symmetrizing form is given by

$$\tau_1(t_w) = \begin{cases} 1 & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let us consider the regular representation of $k\mathfrak{S}_n$. It is the morphism

$$\rho_{\text{reg}} : k\mathfrak{S}_n \to \text{End}_k(k\mathfrak{S}_n)$$

such that for all $(\sigma, \sigma') \in \mathfrak{S}_n$, we have $\rho_{\text{reg}}(t_\sigma t_{\sigma'}) = t_{\sigma\sigma'}$. This implies that its trace is given by

$$\chi_{\text{reg}}(t_w) = \begin{cases} n! & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases}$$

and we thus have $\tau_1 = (1/n!)\chi_{\text{reg}}$. Now it is an easy exercise to show that if the characteristic of k is zero we have $\chi_{\text{reg}} = \sum_{\lambda \in \Pi_n} n^\lambda \chi^\lambda$ where for all $\lambda \in \Pi_n$, the number n^λ denotes the dimension of the associated simple module.

We thus have

$$\tau_1 = (1/n!) \sum_{\lambda \in \Pi_n} n^\lambda \chi^\lambda.$$

This implies that the specialization of the elements $d^\lambda(v)$ at $v = 1$ are nothing but the dimension divided by $n!$!

To summarize, our main problem is solved in the case where the characteristic of k is zero and q is not a root of unity in k (unless $q = 1$): the simple modules are in this case given by the cell modules labeled by the set Π_n of all the partition of n. We moreover know the dimension of these representations. It is given by the number of standard tableaux of the associated form.

What happens in the other cases ? we will still use our remarkable "cell representations". They give rise to non simple modules in general. This means that they have non trivial simple submodules. The idea to solve our main problem will be to study precisely which simple modules appear in our 'cell modules". This leads to the concept of "decomposition matrices" which has been initiated by Brauer for studying the representation theory of finite groups in arbitrary characteristic.

Recall that we have an algebra $\mathcal{H}_k(q)$. To define these new objects, let us consider a cell module V_k^λ labelled by a partition λ of n. As already announced, this module is non simple in general. This means that it has a composition series, that is a series of submodules:

$$0 = M_0 \subset M_1 \subset \ldots \subset M_l = V_k^\lambda,$$

such that the successive quotients M_{i+1}/M_i are simple $\mathcal{H}_k(q)$-modules. If M is a simple module, then we set:

$$[V^\lambda : M] := \sharp\{0 \leq i \leq l-1 \mid M \simeq M_{i+1}/M_i\}.$$

The Jordan-Hölder Theorem asserts that this number is well defined and does not depend on the choice of the composition series. One consequence is that we can compare the dimensions of the cell modules with those of the simple ones.

$$\dim(V^{\lambda}) = \sum_{M\in \mathrm{Irr}(\mathcal{H}_k(q))} [V^{\lambda} : M]\dim(M).$$

We then consider the matrix

$$D_\theta := ([V^{\lambda} : M])_{\lambda\in\Pi_n, M\in\mathrm{Irr}(\mathcal{H}_k(q))}.$$

This is called the decomposition matrix.

We can expect to obtain new information on the simple modules by studying the entries of D_θ. For example, one can easily show using some pure representation theory argument that all simple module appear at least in one of the cell modules as a composition factor. One can also show that this matrix is lower unitriangular with respect to a certain partial order (and thus that the above equations allow the computations of the dimensions of the simple modules) but we will come back later to this point.

Example 4.7. Assume that $n = 3$, $k = \mathbb{Q}(\sqrt{-1})$ and that we have a specialization $\theta : A \to k$ such that $\theta(v) = q$ and such that $q^2 = -1$. Then example 3.12 leads to the following decomposition matrix:

$$\begin{array}{c} (1.1.1) \\ (2.1) \\ (3) \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now let us assume that k_1 is a field of characteristic 3 and that we have a specialization $\theta : A \to k_1$ such that $\theta(v) = 1$. Then, after specialization, Example 3.12 gives us the cell modules. We see that the cell modules over k_1 labeled by (3) and (1.1.1) are simple. This is not the case for the cell module labeled by (2.1). Indeed the associated representation has an invariant subspace generated by $(2, 1) \in k_1^2$. We thus see that this module has a composition series:

$$0 \subset M_1 \subset V_{k_1}^{(2.1)}$$

where $M_1 \simeq V_{k_1}^{(3)}$ and $V_{k_1}^{(2.1)}/M_1 \simeq V_{k_1}^{(1.1.1)}$. We obtain the following decomposition matrix:

$$\begin{array}{c} (1.1.1) \\ (2.1) \\ (3) \end{array} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now let us assume that k_2 is a field of characteristic 2 and that we have a specialization $\theta_2 : A \to k_2$ such that $\theta(v) = 1$. Then example 3.12 leads to the following decomposition matrix:

$$\begin{array}{c} (1.1.1) \\ (2.1) \\ (3) \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We see that the two matrices D_{θ_2} and D_θ are the same (see the next conjecture !).

Before studying in detail this matrix, let us give one of the main motivation for studying the representation theory of Hecke algebras. This is James conjecture [13] which is as follows.

Conjecture 4.8 (James)**.** *Assume that p is a prime number such that $p^2 > n$. Assume that k is a field of characteristic p and let us consider the specialization $\theta_p : \mathbb{Z}[v, v^{-1}] \to k$ sending v to 1. Then we have an associated decomposition matrix D_{θ_p}.*

Assume now that $\theta_0 : \mathbb{Z}[v, v^{-1}] \to \mathbb{Q}(\sqrt{\eta})$ is a specialization sending u to $\eta := exp(2\sqrt{-1}\pi/p))$. Then we have an associated decomposition matrix D_{θ_0}. We have

$$D_{\theta_p} = D_{\theta_0}.$$

This conjecture gives a motivation for the study of the decomposition matrix for the Hecke algebra $\mathcal{H}_{\mathbb{Q}(\eta)}(\eta)$. Indeed, assuming the above conjecture and knowing the decomposition matrix in this case leads to the determination of the dimensions of the simple modules for the symmetric group in characteristic p if $p^2 > n$!

5 Ariki's Theorem

The goal of this section is to give an explicit algorithm for computing the decomposition matrix of the Hecke algebra $\mathcal{H}_k(q)$ over the field $k := \mathbb{Q}(\sqrt{\eta})$ where η is a root of unity of order $e > 1$. As we have seen, we have an associated decomposition matrix which we denote by D_θ (where $\theta : A \to k$ sends v to $\sqrt{\eta}$). This matrix has some important and classical properties which generalize well known properties for finite groups in the modular setting. We refer to [18, Ch. 6] for details.

We have seen that the rows of the decomposition matrix "correspond" to the "decomposition" of the cell modules over the Hecke algebras. We will consider here the columns of this matrix. To do this, let x be an indeterminate and let us consider the $\mathbb{C}(x)$ vector space $\mathcal{F}^n$ generated by the symbols λ with $\lambda \in \Pi_n$. For all $M \in \mathrm{Irr}(\mathcal{H}_k(q))$, we will compute the following elements of $\mathcal{F}^n$:

$$\sum_{\lambda \in \Pi_n} [V^\lambda : M]\lambda$$

They correspond to the columns of $\mathcal{D}_\theta$. Wet set

$$\mathcal{F} := \bigoplus_{n \in \mathbb{N}} \mathcal{F}^n.$$

We need some preliminary results. First, let us define some linear operators for $i = 0, 1, \ldots, e-1$ and $a \in \mathbb{N}$:

$$f_i^{(a)} : \mathcal{F} \to \mathcal{F}.$$

We define the residue of a node $\gamma = (i, j)$ of the Young diagram of a partition λ to be the element $j - i \in \mathbb{Z}/e\mathbb{Z}$. If the node γ can be added to (resp. removed from) $[\lambda]$ so that the new diagram is still a Young diagram of a partition, we say that the associated node is addable (resp. removable). If moreover the residue of γ is i, we say that this is an addable i-node (resp. removable i-node). If μ is a partition of $n+1$ which is obtained from λ by adding a nodes with the same residue i, we denote $\lambda \overset{i:a}{\to} \mu$.

Example 5.1. Set $e = 4$. The following is the Young diagram of $\lambda = (5.3.1.1)$ where each node is filled with the associated residue.

$$\begin{array}{|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 0 \\
\hline
3 & 0 & 1 \\
\cline{1-3}
2 \\
\cline{1-1}
1 \\
\cline{1-1}
\end{array}$$

Note that $(1, 5)$ is a removable 0-node whereas $(2, 3)$ and $(4, 1)$ are both removable 1-nodes. The node $(3, 3)$ is a removable 1-node. These are all the removable nodes of λ. We have 4 addable nodes: $(1, 6)$ which is an addable 1-node, $(2, 4)$ which is an addable 2-node, $(3, 2)$ which is an addable 3-node and $(5, 1)$ which is an addable 0-node.

Let λ and μ be two partitions such that $\lambda \overset{i:a}{\to} \mu$. We set

$$N_i^b(\lambda, \mu) = \sum_{\gamma \in [\mu]/[\lambda]} \sharp\{\text{addable } i-\text{nodes of } \mu \ \text{ below } \gamma\} - \sharp\{\text{removable } i-\text{nodes of } \lambda \ \text{ below } \gamma\}.$$

We can then define the operator $f_i^{(a)}$

$$f_i^{(a)} : \mathcal{F}_n \to \mathcal{F}_{n+a}$$

such that

$$f_i^{(a)}\lambda = \sum_{\lambda \overset{i:a}{\to} \mu} q^{N_i^b(\lambda,\mu)}\mu$$

We consider the $\mathbb{C}(x)$-subspace $\mathcal{G}$ of $\mathcal{F}$ generated by all the elements of the form:

$$f_{i_1}^{(a_1)} \dots f_{i_r}^{(a_r)}\emptyset$$

for all $r \in \mathbb{N}$, $(a_1, \dots, a_r)$ and $i_1, \dots, i_r \in \mathbb{Z}/e\mathbb{Z}$.

We will start by finding a simple basis for this subspace. To do this, let us consider a certain subset of the set of partitions: the set of *e-restricted partitions* which will be denoted by $\mathcal{R}_e$. They are the partitions $(\lambda_1, \dots, \lambda_r)$ such that for all $j = 1, \dots, r-1$, we have $\lambda_i - \lambda_{i+1} < e$.

Example 5.2. The 3-restricted partitions of $n = 5$ are:

$$(1.1.1.1.1), (2.1.1.1), (2.2.1), (3.1.1), (3.2)$$

The easiest way to find a basis of $\mathcal{G}$ is to define elements which are triangular with respect to a certain order on partitions. This partial order is the dominance order and it is defined as follows. Let λ and μ be two partitions of rank n. Then

$$\lambda \trianglelefteq \mu \iff \forall i \in \mathbb{N} \sum_{1 \leq j \leq i} \lambda_j \leq \sum_{1 \leq j \leq i} \mu_j$$

(where the partitions are considered with an infinite number of zero parts). We now define elements $A(\lambda)$'s in $\mathcal{G}$ with $\lambda \in \mathcal{R}_e$ which will give the desired basis. This is done recursively. We define $A(\emptyset) = \emptyset$. Now let $\lambda \in \mathcal{R}_e$ and assume that we have defined $A(\nu$ for all partitions ν with a smaller rank than the one of λ. We construct $A(\lambda) \in \mathcal{G}$ as follows. Consider the lowest removable node of the Young diagram $\gamma_1 = (r, \lambda_r)$ of λ. Assume that its residue is $k \in \mathbb{Z}/e\mathbb{Z}$. Then consider the node $(r-1, \lambda_{r-1})$. If it is a removable k-node then set $\gamma_2 = (r-1, \lambda_{r-1})$, if it is a $k-1$-node then stop the procedure, otherwise consider the node $(r-2, \lambda_{r-2})$ and continue until the procedure stops (which can happen if we meet a $k-1$-node or if we reach the first row). We obtain nodes:

$$\gamma_1, \dots, \gamma_s$$

Consider the partition μ which is obtained by removing all these nodes. By construction, it is a easy to show that this is an e-restricted partition (because λ is) Then we set

$$A(\lambda) = f_k^{(s)} A(\mu)$$

It is an easy combinatorial exercise to show that

$$A(\lambda) = \lambda + \mathbb{C}(x)\text{-linear combination of } \mu \text{ with } \mu \triangleright \lambda$$

We obtain a linearly independent subset of $\mathcal{G}$. Actually, one can show that this is a $\mathbb{C}(x)$-basis.

This basis is not the one that we really want to obtain but it gives an "approximation" of it. Our main basis will satisfy an additional property. It is denoted as follows:

$$\{G(\mu) \mid \mu \in \mathcal{R}_e\}$$

and satisfies the following three conditions:

1. For all $\mu \in \mathcal{R}_e$ and $\lambda \in \Pi_n$ there exist polynomials $d_{\lambda,\mu}(x)$ such that

$$G(\mu) = \sum_{\lambda \in \Pi_n} d_{\lambda,\mu}(x)\lambda.$$

2. We have $d_{\mu,\mu}(x) = 1$ and $d_{\lambda,\mu}(x) = 0$ unless $\lambda \rhd \mu$. In addition, we have $d_{\lambda,\mu}(x) \in x\mathbb{Z}[x]$ if $\lambda \neq \mu$

3. We have
$$G(\mu) = \sum b_{\lambda,\nu}(x) A(\lambda)$$
for elements $b_{\lambda,\nu}(x)$ in $\mathbb{Z}[x, x^{-1}]$ with $b_{\lambda,\mu}(x^{-1}) = b_{\lambda,\mu}(x)$.

There is an easy and purely combinatorial way to find this basis from our first basis. We refer to [18] for details.

Example 5.3. We give an example for the constructions of the canonical basis for $n = 5$ and $e = 2$. We have three 2-restricted partitions:
$$(1.1.1.1.1),\ (2.1.1.1),\ (2.2.1)$$
Let us first consider $\lambda = (2.2.1)$.

0	1
1	0
0	

We have $A(\lambda) = f_0^{(2)} f_1^{(2)} f_0 \emptyset$. We see that:
$$A(\lambda) = (2.2.1) + x(3.1.1) + x^2(3.2)$$
Similarly, one can check that
$$A(2.1.1.1) = (2.1.1.1) + x(4.1)$$
By definition, these two elements are $G(2.2.1)$ and $G(2.1.1.1)$ respectively. Now
$$A(1.1.1.1.1) = (1.1.1.1.1) + (2.2.1) + 2x(3.1.1) + x^2(3.2) + x^2(5)$$
It is not the element $G(1.1.1.1.1)$ because (2) is not satisfied. However we have
$$A(1.1.1.1.1) - A(2.2.1) = (1.1.1.1.1) + x(3.1.1) + x^2(5)$$
which is by definition $G(1.1.1.1.1)$.

We can now state the main theorem of this section which gives the goal of all these combinatorial studies in terms of the representation theory of Hecke algebras.

Theorem 5.4 (Ariki [1], Lascoux-Leclerc-Thibon's conjecture [17])**.** *Assume that $\theta : A \to \mathbb{C}$ is a specialization such that $\theta(v^2) = \eta$ is a primitive root of unity of order e. Let D_θ be the decomposition matrix associated to the resulting Hecke algebra $\mathcal{H}_{\mathbb{C}}(\eta)$. Let us consider the matrix*
$$D_e(x) = (d_{\lambda,\mu}(x))_{\lambda \in \Pi_n, \mu \in \mathcal{R}_e}.$$
Then we have
$$D_e(1) = D_\theta.$$

Remark 5.5. This theorem and the above algorithm admit a generalization for a larger class of complex reflection groups: the wreath product of the symmetric group with a cyclic group (see also [1] and [11] for the computation of the associated decomposition matrices).

The proof of this theorem requires a large number of sophisticated results using affine Hecke algebras, intersection cohomology and representations of affine Lie algebras. In fact, the space $\mathcal{F}$ and the subspace $\mathcal{G}$ that we have constructed above have the structure of modules over the quantum group in affine type A. The basis that leads to the decomposition matrix corresponds to a well known basis in the representation theory of quantum groups, the Kashiwara-Lusztig canonical basis.

As a consequence, we obtain an algorithm for the computation of the decomposition matrices for the Hecke algebra in characteristic 0. Note also that this Theorem yields that the simple modules for $\mathcal{H}_k(\eta)$ are naturally indexed by the e-restricted partitions. Indeed the above theorem implies that the decomposition matrix has the following form:

$$D_\theta = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] \mathcal{R}_e \\ \\ \\ \\ \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\right] \Pi_n$$

As the columns of the decomposition matrix are indexed by the simple $\mathcal{H}_k(\eta)$-modules, we have a canonical bijection between these modules and the set $\mathcal{R}_e$. This was already known by the results of Dipper and James [5].

Example 5.6. Assume that $n = 3$ and $e = 2$. Then one can compute the elements $A(1.1.1)$ and $A(2.1)$. We have:

$$A(1.1.1) = G(1.1.1) = f_0 f_1 f_0.\emptyset = (1.1.1) + x(3) \text{ and } A(2.1) = G(2.1) = f_1^{(2)} f_0.\emptyset = (2.1).$$

Thus, the matrix $D_e(x)$ is

$$\begin{array}{c} (1.1.1) \\ (2.1) \\ (3) \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x & 0 \end{pmatrix}$$

and we can check Theorem 5.4 using Example 4.7.

6 Quantized decomposition matrices

The study of the representation theory of Hecke algebras leads to several exciting new developments thanks to the works of Khovanov, Lauda [8], Rouquier [19], Brundan and Kleshchev [3]. The starting point is the following problem: Ariki's Theorem asserts that the decomposition matrix for the Hecke algebra (over a field of characteristic 0) is given by the matrix $D_e(x)$ specialized at $x = 1$. Could it be that the non specialized matrix has also an interpretation in the context of the representation theory of Hecke algebras ? In all this part, we consider the Hecke algebra $\mathcal{H}_k(q)$ over an arbitrary field k. A nice and complete survey on this part is given in [15].

One of the starting points in order to answer to that question is to show that our Hecke algebra is endowed with a grading. To do this, one can construct 3 types of remarkable elements in the Hecke algebra:

- a certain set of orthogonal idempotents
$$\{e(\underline{i}) \mid \underline{i} \in (\mathbb{Z}/e\mathbb{Z})^n\}$$
- a certain set of nilpotent elements:
$$\{y_i \mid i = 1, \ldots, n\}$$
- and certain "intertwining" elements:
$$\{\psi_j \mid i = 1, \ldots, n\}$$

Theorem 6.1 (Brundan-Kleshchev [3])**.** *The above generators satisfy the relations of a graded algebra called the (modified) KLR algebra. The Hecke algebra $\mathcal{H}_k(q)$ is isomorphic to this algebra.*

Thus, one immediate consequence of this result is that $\mathcal{H}_k(q)$ is graded over k. This means that $\mathcal{H}_k(q)$ decomposes into a direct sum of vector spaces

$$\mathcal{H}_k(q) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}_k(q)_i.$$

such that for all $(i,j)\in\mathbb{Z}^2$ we have

$$\mathcal{H}_k(q)_i\mathcal{H}_k(q)_j \subset \mathcal{H}_k(q)_{i+j}$$

Another nice and useful consequence of the above theorem is the following. Set

$$e := \min\{i>0 \mid 1+q+q^2+\ldots+q^{i-1}\} \in \mathbb{N}\cup\{\infty\}$$

Then, looking at the presentation of the KLR algebras, we see immediately that the algebra does not really depend on q but rather on e.

Now that we know that this algebra is graded we can try to study its *graded representation theory.* This means that we look at at the graded modules of the Hecke algebras. Those are the $\mathcal{H}_k(q)$-modules M which have a decomposition into vector spaces $M=\bigoplus_{i\in\mathbb{Z}} M_i$ such that $\mathcal{H}_k(q)_i M_j \subset M_{i+j}$. For such a module, one can construct its graded shifted, that is the $\mathcal{H}_k(q)$-module $M\langle m\rangle$ which is obtained by shifting the grading up by m: $M\langle m\rangle_n = M_{n-m}$.

It is a classical result of graded representation theory to show that the simple $\mathcal{H}_k(q)$-modules are endowed with a unique canonical grading (we have to do that to fix a grading so that the module is autodual with respect to a certain automorphism). In addition, all the simple graded modules are given by these simples and by their shifted, up to graded isomorphism (that is, isomorphism which respect the gradings)

Now assume that the characteristic of k is zero and that q is a primitive root of unity of order $e>1$.Then we have our Hecke algebra $\mathcal{H}_k(q)$. We will not work here with the cell modules that we have introduced above but rather with another type of modules which look like them: the Specht modules introduced by Dipper, James and Mathas using a different basis than the Kazhdan-Lusztig basis, [6] (see also [16] for the relations between the cell modules and the different version of Specht modules). They are still labelled by the set of partitions of n (and are isomorphic to the cell modules in the semi simple case.) Let

$$\{S^\lambda \mid \lambda\in\Pi_n\}.$$

be the set of Specht modules for $\mathcal{H}_k(q)$. Brundan, Kleshchev and Wang [4] have shown that these modules are also endowed with a canonical grading. Using the same idea as for the introduction of the decomposition matrix,iIt makes sense to consider the graded composition series of these graded Specht modules, that is, the sequence of graded submodules (that is submodule which respect the graduation) such that the successive quotient are graded simple. We can then define the *graded decomposition numbers* which are polynomials in an indeterminate x:

$$[S^\lambda : M]_x := \sum_{n\in\mathbb{N}} a_n x^n$$

for all $M\in\mathrm{Irr}(\mathcal{H}_k(q))$ where a_n denotes the multiplicity of $M\langle n\rangle$ in a graded composition series of S^λ (A Jordan-Hölder Theorem still holds in the graded setting). We can thus define the graded decomposition matrix by

$$\mathcal{D}_e(x) = ([S^\lambda : M]_x)_{\lambda\in\Pi_n, M\in\mathrm{Irr}(\mathcal{H}_k(q))}$$

If we set $x=1$, we can see that we recover the ordinary decomposition matrix

Theorem 6.2 (Brundan-Kleshchev)**.** *Recall the matrix $D_e(x)$ in Theorem 5.4. Then, under the above assumptions we have:*

$$\mathcal{D}_e(x) = D_e(x)$$

This theorem thus gives an interpretation of the matrix of the canonical basis, and a graded (or quantum) analogue of Ariki's Theorem.

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